Saturating Constructions for Normed Spaces II

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Abstract

We prove several results of the following type: given finite dimensional normed space V possessing certain geometric property there exists another space X having the same property and such that (1) $\log \dim X = O(\log \dim V)$ and (2) every subspace of X, whose dimension is not "too small," contains a further well-complemented subspace nearly isometric to V. This sheds new light on the structure of large subspaces or quotients of normed spaces (resp., large sections or linear images of convex bodies) and provides definitive solutions to several problems stated in the 1980s by V. Milman.

1 Introduction

This paper continues the study of the *saturation phenomenon* that was discovered in [ST] and of the effect it has on our understanding of the structure of high-dimensional normed spaces and convex bodies. In particular, we obtain here a dichotomy-type result which offers a near definitive treatment of some aspects of the phenomenon. We sketch first some background ideas and hint on the broader motivation explaining the interest in the subject.

Much of geometric functional analysis revolves around the study of the family of subspaces (or, dually, of quotients) of a given Banach space. In the finite dimensional case this has a clear geometric interpretation: a normed space is determined by its unit ball, a centrally symmetric convex body, subspaces correspond to sections of that body, and quotients to projections (or, more generally, linear images). Such considerations are very natural from the geometric or linear-algebraic point of view, but they also have a

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bearing on much more applied matters. For example, a convex set may represent all possible states of a physical system, and its sections or images may be related to approximation or encoding schemes, or to results of an experiment performed on the system. It is thus vital to know to what degree the structure of the entire space (resp., the entire set) can be recovered from the knowledge of its subspaces or quotients (resp., sections/images). At the same time, one wants to detect some possible regularities in the structure of subspaces which might have not existed in the whole space.

A seminal result in this direction is the 1961 Dvoretzky theorem, with the 1971 strengthening due to Milman, which says that every symmetric convex body of large dimension n admits central sections which are approximately ellipsoidal and whose dimension k is of order $\log n$ (the order that is, in general, optimal). Another major result was the discovery of Milman [M2] from the mid 1980's that *every* n-dimensional normed space admits a *subspace of a quotient* which is "nearly" Euclidean and whose dimension is $\geq \theta n$, where $\theta \in (0,1)$ is arbitrary (with the exact meaning of "nearly" depending only on θ). Moreover, a byproduct of the approach from [M2] was the fact that every n-dimensional normed space admits a "proportional dimensional" quotient of bounded *volume ratio*, a volumetric characteristic of a body closely related to cotype properties (we refer to [MS1], [T] and [P2] for definitions of these and other basic notions and results that are relevant here). This showed that one can get a very essential regularity in a global invariant of a space by passing to a quotient *or* a subspace of dimension, say, approximately $n/2$. It was thus natural to ask whether similar statements may be true for other related characteristics. This line of thinking was exemplified in a series of problems posed by Milman in his 1986 ICM Berkeley lecture [M3].

The paper [ST] elucidated this circle of ideas and, in particular, answered some of the problems from [M3]. A special but archetypal case of the main theorem from [ST] showed the existence of an n-dimensional space Y whose *every* subspace (resp., *every* quotient) of dimension $\geq n/2$ contains a further 1-complemented subspace isometric to a preassigned (but a priori arbitrary) k-dimensional space V, as long as k is at most of order $\sqrt{n}$. In a sense, Y was *saturated* with copies of the V. This led to the discovery of the following phenomenon: passing to large subspaces or quotients *can not*, in general, erase k-dimensional features of a space if k is below certain threshold value depending on the dimension of the initial space and the exact meaning of "large." In the particular case stated above, i.e., that of "proportional" subspaces or quotients, the threshold dimension was (at least) of order $\sqrt{n}$,

and "impossibility to erase" meant that every such subspace (resp., quotient map) preserved a copy of the given V.

However, the methods presented in [ST] were not sufficient for a definitive treatment of the issue at hand. For example, we prove in the present paper that, for any $q > 2$, there are spaces of cotype q (of arbitrarily high dimension n, with uniform control of constants) whose all, say, $n/2$-dimensional subspaces are poorly K-convex (or, equivalently, contain rather large subspaces well-isomorphic to finite-dimensional ℓ_1-spaces). This is in stark contrast to the extremal case of $q = 2$: as it has been known since mid 1970's, every space of cotype 2 admits proportional subspaces which are nearly Euclidean (which is of course incomparably stronger than being K-convex). By comparison, in [ST] a similar result was established only for $q > 4$. This answered one of the questions of Milman, but still left open a possibility that an intermediate hypothesis weaker than cotype 2 (such as cotype q with $2 < q \leq 4$) could force existence of nice subspaces. Our present theorem closes this gap completely, and has the character of a dichotomy: for $q = 2$ every space of cotype 2 admits proportional nearly Euclidean subspaces, while for any $q > 2$ there exist spaces of cotype q without large K-convex subspaces at all. It was important to clarify this point since hypothetical intermediate threshold values of q (namely, $q = 4$) appeared in related – and still not completely explained – contexts in the asymptotic geometric analysis literature, cf. [B] (see also Proposition 27.5 in [T]) or [P1].

Another variation of the saturation phenomenon that is being considered here addresses what has being referred to recently as "global properties." It has been realized in the last few years (cf. [MS2]) that many *local* phenomena (i.e., referring to subspaces or quotients of a normed space) have *global* analogues, expressed in terms of the entire space. For example, a "proportional" quotient of a normed space corresponds to the Minkowski sum of several rotations of its unit ball. Dually, a "proportional" subspace corresponds to the intersection of several rotations. (Such results were already implicit, e.g., in [K].) Here we prove a sample theorem in this direction concerning the Minkowski sum of two rotations of a unit ball, which answers a query directed to us by V. Milman.

We use the probabilistic method, and employ the "blueprint" for constructing random spaces that was developed by Gluskin in [G] (the reader is also referred to [MT] for a survey of other results and methods in this direction). In their most general outline, our arguments parallel those of [ST]. However, there are substantial differences, and the present considerations are

much more subtle than those of [ST]. Moreover, we believe that several ingredients (such as a usage of Lemma 3.2-like statement to enable decoupling of otherwise dependent events, or Lemma 3.3), while playing mostly technical role in this paper, are sufficiently fundamental to be of independent interest.

The organization of the paper is as follows. In the next section we describe our main results and their immediate consequences. We also explain there the needed conventions employed by experts in the field, but not necessarily familiar to the more general mathematical reader. (Otherwise, we use the standard notation of convexity and geometric functional analysis as can be found, e.g., in [MS1], [P2] or [T].) Section 3 contains the proof of Theorem 2.1, relevant to the dichotomy mentioned above and to Problems 1-3 from [M3]. Section 4 deals with the global variant of the saturation phenomenon.

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2 Description of results

The first result we describe is a *subspace* saturation theorem. The approach of [ST] makes it easy to implement a saturation property for subspaces. Indeed, the dual space X^* of the space constructed in [ST], Theorem 2.1 has the property that, under some assumptions on m, k and $n := \dim X^*$, every m dimensional *subspace* of X^* contains a (1-complemented) subspace isometric to V (where V is a preassigned k-dimensional space). In this paper we show that the construction can be performed while preserving geometric features of the space V (specifically, cotype properties), a trait which is crucial to applications.

Theorem 2.1 *Let $q \in (2, \infty]$ and let $\varepsilon > 0$. Then there exist $\alpha = \alpha_q \in (0, 1)$ and $c = c_{q,\varepsilon} > 0$ such that whenever positive integers n and m_0 verify $c^{-1} n^\alpha \le m_0 \le n$ and V is* **any** *normed space with*

$$\dim V \le c m_0 / n^\alpha,$$

then there exists an n-dimensional normed space Y whose cotype q constant is bounded by a function of q and the cotype q constant of V and such that,

for any $m_0 \le m \le n$, **every** m*-dimensional subspace* $\tilde{Y}$ *of* Y *contains a* $(1+\varepsilon)$*-complemented subspace* $(1+\varepsilon)$*-isomorphic to* V*.*

Let us start with several remarks concerning the hypotheses on $k := \dim V$ and m_0 included in the statement above. If, say, $m_0 \approx n/2$, then k of order "almost" $n^{1-\alpha}$ is allowed. Nontrivial (i.e., large) values of k are obtained whenever $m_0 \gg n^{\alpha}$; we included the lower bound on m_0 in the statement to indicate for which values of the parameters the assertion of the Theorem is meaningful.

We can now comment on the relevance of Theorem 2.1 to problems from [M3]. Roughly speaking, Problems 2 and 3 asked whether every space of nontrivial cotype $q < \infty$ contains a proportional subspace of type 2, or even just K-convex. This is well known to be true if $q = 2$ due to presence of nearly Euclidean subspaces [For a reader not familiar with the type/cotype theory it will be "almost" sufficient to know that a nontrivial (i.e., finite) cotype property of a space is equivalent to the absence of large subspaces well-isomorphic to ℓ_∞-spaces; similarly, nontrivial type properties and K-convexity are related to the absence of ℓ_1-subspaces.] Accordingly, by choosing, for example, $V = \ell_1^k$ in the Theorem, we obtain – in view of the remarks in the preceding paragraph on the allowed values of k and m – a space whose all "large" subspaces contain isometrically ℓ_1^k and which consequently provides a counterexample to the problems for *any* $q > 2$. More precisely, if m_0 is "proportional" to n and $V = \ell_1^k$ is of the maximal dimension that is allowed, then the type 2 constant of any corresponding subspace $\tilde{Y}$ of Y from the Theorem is at least of order $n^{(1-\alpha)/2}$ (and analogously for any nontrivial type $p > 1$). The K-convexity constant of any such $\tilde{Y}$ is at least of order $\sqrt{\log n}$ (up to a constant depending on q). Problems 2 and 3 from [M3] are thus answered in the negative in a very strong sense. Problem 1 from [M3] corresponds to $q = \infty$ in Theorem 2.1 (i.e., no cotype assumptions) and has already been satisfactorily treated in [ST]; however, the present paper offers a unified discussion of all the issues involved (see also related comments later in this section).

We also remark that choosing $V = \ell_p^k$ (for some $1 < p < 2$) in Theorem 2.1 leads to a space Y whose type p and cotype q constants are bounded by numerical constants and such that, for every m-dimensional subspace $\tilde{Y}$ of Y and every $p < p_1 < 2$, the type p_1 constant of $\tilde{Y}$ is at least $k^{1/p-1/p_1}$. If m_0 is "proportional" to n, the type p_1 constant of $\tilde{Y}$ is at least of order $n^{(1-\alpha)(1/p-1/p_1)}$, in particular it tends to $+\infty$ as $n \to \infty$. On the other hand,

the spaces $\tilde{Y}$ and Y are then, by construction, uniformly (in n) K-convex.

Theorem 2.1 will be an immediate consequence of the more precise and more technical Proposition 3.1 stated in the next section. That statement makes the dependence of the parameters c, α on $\varepsilon > 0$ and $q \in (2, \infty)$ more explicit. This will allow us, by letting $q \to \infty$, to retrieve the case $q = \infty$ and then, by passing to dual spaces, to reconstruct (up to a logarithmic factor) the main theorem from [ST]: *if n, m_0 and k satisfy $\sqrt{n \log n} \le m_0 \le n$ and $k \le m_0/\sqrt{n \log n}$, then for every k-dimensional normed space W there exists an n-dimensional normed space X such that every quotient $\tilde{X}$ of X with* $\dim \tilde{X} \ge m_0$ *contains a 1-complemented subspace isometric to W.*

We wish now to offer a few comments on the construction that is behind Theorem 2.1, and which is implicit in Proposition 3.1. To this end, we recall some notation and sketch certain ideas from [ST], which also underlie the present argument.

If W is a normed space and $1 \le p < \infty$, by $\ell_p^N(W)$ we denote the ℓ_p-sum of N copies of W, that is, the space of N-tuples $(x_1, \dots, x_N)$ with $x_i \in W$ for $1 \le i \le N$, with the norm $\|(x_1, \dots, x_N)\| = (\sum_i \|x_i\|^p)^{1/p}$. It is a fundamental and well-known fact that the spaces $\ell_p^N(W)$ inherit type and cotype properties of the space W, in the appropriate ranges of p (cf. e.g., [T], §4).

The saturating construction from [ST] obtained X^* as a (random) subspace of $\ell_\infty^N(V)$, for appropriate value of N. This is not the right course of action in the context of Theorem 2.1 since such a subspace will typically contain rather large subspaces well-isomorphic to ℓ_∞^s, hence failing to possess any nontrivial cotype property. However, substituting q for ∞ works: the space $\ell_q^N(V)$ and all its subspaces will be of cotype q if V is. The approach of [ST] was to concentrate on the case of $\ell_\infty^N(V)$, and then to use the available "margin of error" to transfer the results to q sufficiently close to ∞. By contrast, to handle the entire range $2 < q < \infty$ we need to work *directly* in the ℓ_q setting, which – as is well known to analysts – often requires much more subtle considerations.

To state the next theorem, it will be helpful to subscribe to the following "philosophy" and notational conventions. Since a normed space X is completely described by its unit ball $K = B_X$ or its norm $\|\cdot\|_X$, we shall tend to identify these three objects. In particular, we will write $\|\cdot\|_K$ for the Minkowski functional defined by a centrally symmetric convex body $K \subset \mathbb{R}^n$ and denote the resulting normed space by $(\mathbb{R}^n, \|\cdot\|_K)$ or just $(\mathbb{R}^n, K)$. Two

normed spaces are isometric iff the corresponding convex bodies are affinely equivalent.

As suggested in the Introduction, it is of interest to consider "global" analogues of Theorem 2.1-like statements. The following is a sample result that corresponds to the "local" Theorem 2.1 of [ST], and that was already announced in that paper.

Theorem 2.2 *There exists a constant $c > 0$ such that, for any positive integers n, k satisfying $k \leq cn^{1/4}$ and for every k-dimensional normed space W, there exists an n-dimensional normed space $X = (\mathbb{R}^n, K)$ such that, for any $u \in O(n)$, the normed space $(\mathbb{R}^n, K + u(K))$ contains a 3-complemented subspace 3-isomorphic to W.*

In general, the interplay between the global and local results is not fully understood. While it is an experimental fact that a parallel between the two settings exists, there is no formal conceptual framework which explains it. It is thus important to provide more examples in hope of clarifying the connection. It is also an experimental fact that the local results and their global analogues sometimes vary in difficulty. In the present context, the proof of Theorem 2.2 is substantially more involved than that of its local counterpart, Theorem 2.1 from [ST].

We conclude this section with several comments about notation. As mentioned earlier, our terminology is standard in the field and all unexplained concepts and notation can be found, e.g., in [MS1], [P2] or [T]. The standard Euclidean norm on $\mathbb{R}^n$ will be always denoted by $|\cdot|$. (*Attention:* the same notation may mean elsewhere cardinality of a set and, of course, the absolute value of a scalar.) We will write B_2^n for the unit ball in ℓ_2^n and, similarly but less frequently, B_p^n for the unit ball in ℓ_p^n, $1 \leq p \leq \infty$.

For a set $S \subset \mathbb{R}^n$, by $\mathrm{conv}\,(S)$ we denote the convex hull of S. If $1 \leq p < \infty$, we denote by $\mathrm{conv}_p(S)$ the p-convex hull of S, that is, the set of vectors of the form $\sum_i t_i x_i$, where $t_i > 0$ and $x_i \in S$ for all i, and $\sum_i t_i^p = 1$. (In particular, for $p = 1$, $\mathrm{conv}_p(S) = \mathrm{conv}\,(S)$.)

The arguments below will use various subsets of $\mathbb{R}^n$ obtained as convex hulls or p-convex hulls, for $1 < p < \infty$, of some more elementary sets, or linear images of those; indeed for Theorem 2.1 we have to consider the case of $p > 1$, while in Theorem 2.2 the case of $p = 1$ is sufficient. In order to emphasize the parallel roles which these sets (and other objects) play in the

proofs (which is also closely related to the role they play in [ST]), we try to keep a fully analogous notation for them, and to distinguish them by adding a subscript $\cdot_p$ when the set depends on p.

3 Saturating spaces of cotype $q > 2$

Theorem 2.1 will be an immediate consequence of the following technical proposition.

Proposition 3.1 *Let $2 < q < \infty$ and set $\alpha := (q-2)/(2q+2)$ $(\in (0, 1/2))$. Let n and m_0 be positive integers with $\sqrt{q}\, n^{1-\alpha}(\log n)^{(1-2\alpha)/3} \le m_0 \le n$. Let V be any normed space with*

$$\dim V \le \frac{c_1 m_0}{q^{1/2}\ n^{1-\alpha}(\log n)^{(1-2\alpha)/3}}$$

(where $c_1 > 0$ is an appropriate universal constant). Then there exists an n-dimensional normed space Y whose cotype q constant is bounded by a function of q and the cotype q constant of V and such that, for any $m_0 \le m \le n$, every m-dimensional subspace $\tilde{Y}$ of Y contains a $2^{1/q}$-complemented subspace $2^{1/q}$-isomorphic to V. Moreover, for every $\varepsilon > 0$, we may replace the quantity $2^{1/q}$ by $1 + \varepsilon$, at the cost of allowing c_1 to depend on ε.

Proof Fix $2 < q < \infty$ and let $p = q/(q-1)$ be the conjugate exponent. Let $1 \le k \le m \le n \le kN$ be positive integers. More restrictions will be added on these parameters as we proceed, and in particular we shall specify N (depending also on q) at the end of the proof. Notice that choosing the constant c_1 small makes the assertion vacuously satisfied for small values of m_0, and so we may and shall assume that m_0, n and N are large.

Let V be a k-dimensional normed space. Identify V with $\mathbb{R}^k$ in such a way that the Euclidean ball B_2^k and the unit ball B_V of V satisfy $B_2^k \subset B_V \subset \sqrt{k}\, B_2^k$ (for example, B_2^k may be the ellipsoid of maximal volume contained in B_V). As indicated in the preceding section, we shall construct the space Y as a (random) subspace of $\ell_q^N(V)$, the ℓ_q-sum of N copies of V. We will actually work in the dual setting of random quotients of $Z_p := \ell_p^N(W)$, where $W := V^*$; as frequent in this type of constructions, the geometry of that setting is more transparent. The above identification of V with $\mathbb{R}^k$ induces the identification of W with $\mathbb{R}^k$, and thus allows to identify Z_p with $\mathbb{R}^{Nk}$.

Let $G = G(\omega)$ be a $n \times Nk$ random matrix (defined on some underlying probability space $(\Omega, \mathbb{P})$) with independent $N(0, 1/n)$-distributed Gaussian entries. Consider G as a linear operator $G : \mathbb{R}^{Nk} \to \mathbb{R}^n$ and set

$$K_p = B_{X_p(\omega)} := G(\omega)(B_{Z_p}) \subset \mathbb{R}^n. \tag{3.1}$$

The random normed space $X_p = X_p(\omega)$ can be thought of as a random (Gaussian) quotient of Z_p, with $G(\omega)$ the corresponding quotient map and K_p the unit ball of X_p. [The normalization of G is not important; here we choose it so that, with k, N in the ranges that matter, the radius of the Euclidean ball circumscribed on K_p be typically comparable to 1.]

We reiterate that the dual spaces $X_p^* = X_p(\omega)^*$ are isometric to subspaces of $Z_p^* = \ell_q^N(V)$ and so their cotype q constants are uniformly bounded (depending on q and the cotype q constant of V). We shall show that, for appropriate choices of the parameters, the space $Y = X_p(\omega)^*$ satisfies, with probability close to 1, the (remaining) assertion of Theorem 2.1 involving the subspaces well-isomorphic to V. This will follow if we show that, outside of a small exceptional set, every quotient $\tilde{X}_p(\omega)$ of $X_p(\omega)$ of dimension $m \geq m_0$ contains a $2^{1/q}$-complemented subspace $2^{1/q}$-isomorphic to W, for values of k described in Theorem 2.1 (and analogously for $1 + \varepsilon$ in place of $2^{1/q}$). To be absolutely precise, we shall show that the identity on W well factors through $X_p(\omega)$, a property which dualizes without any loss of the constant involved. Thus we have a very similar problem to the one considered in [ST], however the present context requires several subtle technical modifications of the argument applied there.

Similarly as in [ST], we will follow the scheme first employed in [G]: *Step I* showing that the assertion of the theorem is satisfied for a *fixed* quotient map with probability close to 1; *Step II* showing that the assertion is "essentially stable" under small perturbations of the quotient map; and *Step III* which involves a discretization argument.

We start by introducing some notation that will be used throughout the paper. Denote by $F_1, \ldots, F_N$ the k-dimensional coordinate subspaces of $\mathbb{R}^{Nk}$ corresponding to the consecutive copies of W in Z_p. In particular, from the definition of the ℓ_p-sum we have

$$B_{Z_p} = \mathrm{conv}_p(F_j \cap B_{Z_p} : j \in \{1, \ldots, N\}).$$

For $j = 1, \ldots, N$, we define subsets of $\mathbb{R}^n$ as follows: $E_j := G(F_j)$, $K_j := G(F_j \cap B_{Z_p})$ and

$$K'_{j,p} := G(\mathrm{span}\,[F_i : i \neq j] \cap B_{Z_p}). = \mathrm{conv}_p(K_i : i \neq j). \tag{3.2}$$

We point out certain ambiguity in the notation: K_p, $p \in (1,2)$, is the unit ball of X_p, while K_j, $j \in \{1,\ldots,N\}$ stands for the section of K_p corresponding to E_j. This should not lead to confusion since, first, the sections *do not* depend on p and, second, p remains fixed throughout the argument. (Similar caveats apply to the families of sets $D.$, $\tilde{K}.$ and $\tilde{D}.$ which are defined in what follows.)

In addition to K_p and the K_j's, we shall need subsets constructed in an analogous way from the Euclidean balls. First, for $j = 1,\ldots,N$, set $D_j := G(F_j \cap B_2^{Nk})$. Then let

$$D_p := G\left(\mathrm{conv}_p(F_j \cap B_2^{Nk} : j \in \{1,\ldots,N\})\right) = \mathrm{conv}_p(D_j : j \in \{1,\ldots,N\}). \tag{3.3}$$

Next, for $j = 1,\ldots,N$, let

$$D'_{j,p} := G\left(\mathrm{conv}_p(F_i \cap B_2^{Nk} : i \neq j)\right) = \mathrm{conv}_p(D_i : i \neq j). \tag{3.4}$$

Finally, for a subset $I \subset \{1,\ldots,N\}$, we let

$$D_{I,p} := G\left(\mathrm{conv}_p(F_i \cap B_2^{Nk} : i \in I)\right) = \mathrm{conv}_p(D_i : i \in I). \tag{3.5}$$

Note that since $\frac{1}{\sqrt{k}}B_2^k \subset B_W \subset B_2^k$, it follows that $\frac{1}{\sqrt{k}}D_j \subset K_j \subset D_j$. Consequently, analogous inclusions hold for all the corresponding K- and D-type sets as they are p-convex hulls of the appropriate K_j's and D_j's.

Step I. Analysis of a single quotient map. Since a quotient space is determined up to an isometry by the kernel of a quotient map, it is enough to consider quotient maps which are *orthogonal* projections. Let, for the time being, $Q : \mathbb{R}^n \to \mathbb{R}^m$ be the canonical projection on the first m coordinates. In view of symmetries of our probabilistic model, all relevant features of this special case will transfer to an arbitrary rank m orthogonal projection.

Let $\tilde{G} = QG$, i.e., $\tilde{G}$ is the $m \times Nk$ Gaussian matrix obtained by restricting G to the first m rows. Let $\tilde{K}_p = Q(K_p) = \tilde{G}(B_{Z_p})$ and denote the space $(\mathbb{R}^m, \tilde{K}_p)$ by $\tilde{X}_p$; the space $\tilde{X}_p$ is the quotient of X_p induced by the quotient map Q. We shall use the notation of $\tilde{E}_j$, $\tilde{K}_j$, $\tilde{K}'_{j,p}$ for the subsets of $\mathbb{R}^m$ defined in the same way as E_j, K_j, $K'_{j,p}$, above, but using the matrix $\tilde{G}$ in place of G. Analogous convention is used to define the $\tilde{D}$-type sets $\tilde{D}_p$, $\tilde{D}_j$ and $\tilde{D}'_{j,p}$.

For any subspace $H \subset \mathbb{R}^m$, we will denote by P_H the orthogonal projection onto H. We shall show that outside of an exceptional set of small measure there exists $j \in \{1,\ldots,N\}$ such that $P_{\tilde{E}_j}(\tilde{K}'_{j,p}) \subset \tilde{K}_j$. Note that,

for any given i, we *always* have $\tilde{K}_p = \mathrm{conv}_p(\tilde{K}_i, \tilde{K}'_{i,p})$ and $\tilde{K}_i \subset \tilde{E}_i$. It follows that, for j as above,

$$P_{\tilde{E}_j}(\tilde{K}_p) = \mathrm{conv}_p(\tilde{K}_j, P_{\tilde{E}_j}(\tilde{K}'_{j,p})) \subset 2^{1/q}\tilde{K}_j. \tag{3.6}$$

Note that $\tilde{K}_j$ is an affine image of the ball $F_j \cap B_{Z_p}$, which is the ball B_W on coordinates from F_j. On the other hand, $\tilde{E}_j$ considered as a subspace of $\tilde{X}_p$ (thus endowed with the ball $\tilde{E}_j \cap \tilde{K}_p$) satisfies, by (3.6), $\tilde{K}_j \subset \tilde{E}_j \cap \tilde{K}_p \subset 2^{1/q}\tilde{K}_j$, which makes it $2^{1/q}$-isomorphic to B_W. Using (3.6) again we also get the $2^{1/q}$-complementation. (Similarly, $P_{\tilde{E}_j}(\tilde{K}'_{j,p}) \subset \varepsilon\tilde{K}_j$ will imply $(1+\varepsilon)$-isomorphism and $(1+\varepsilon)$-complementation.)

Returning to inclusions between the K- and D-type sets, they also hold for the $\tilde{K}$- and $\tilde{D}$-type sets, so that, for example, $\frac{1}{\sqrt{k}}\tilde{D}_j \subset \tilde{K}_j \subset \tilde{D}_j$. Consequently, in order for the inclusion $P_{\tilde{E}_j}(\tilde{K}'_{j,p}) \subset \tilde{K}_j$ to hold it is enough to have

$$P_{\tilde{E}_j}(\tilde{D}'_{j,p}) \subset \frac{1}{\sqrt{k}}\tilde{D}_j. \tag{3.7}$$

The rest of the proof of Step I is to show that, with an appropriate choice of the parameters, this seemingly rough condition is satisfied for some $1 \le j \le N$, outside of a small exceptional set.

Let us now pass to the definition of the exceptional set. We start by introducing, for $j \in \{1, \dots, N\}$, the "good" sets. Fix a parameter $0 < \kappa \le 1$ to be determined later, and let

$$\Theta'_j := \left\{\omega \in \Omega : P_{\tilde{E}_j}(\tilde{D}'_{j,p}) \subset \kappa B_2^m\right\} \tag{3.8}$$

$$\Theta'_{j,0} := \left\{\frac{1}{2}\sqrt{\frac{m}{n}}(B_2^m \cap \tilde{E}_j) \subset \tilde{D}_j \subset 2\sqrt{\frac{m}{n}}(B_2^m \cap \tilde{E}_j)\right\}. \tag{3.9}$$

Now if κ, k, m and n satisfy

$$\kappa \le \frac{1}{\sqrt{k}} \cdot \frac{1}{2}\sqrt{\frac{m}{n}}, \tag{3.10}$$

then, for $\omega \in \Theta'_j \cap \Theta'_{j,0}$, the inclusion (3.7) holds. Thus, outside of the exceptional set

$$\Theta^0 := \Omega \setminus \bigcup_{1\le j\le N} \left(\Theta'_j \cap \Theta'_{j,0}\right) = \bigcap_{1\le j\le N} \left((\Omega \setminus \Theta'_j) \cup (\Omega \setminus \Theta'_{j,0})\right) \tag{3.11}$$

there exists $j \in \{1, \ldots, N\}$ such that (3.7) holds, and this implies, by an earlier argument, that there exists $j \in \{1, \ldots, N\}$ such that $\tilde{E}_j$ considered as a subspace of $\tilde{X}_p$ is $2^{1/q}$-complemented and $2^{1/q}$-isomorphic to W.

It remains to show that the measure of the exceptional set Θ^0 is appropriately small; this will be the most technical part of the argument. The first problem we face is that the events entering the definition of Θ^0 are not independent as j varies. We overcome this difficulty by a decoupling trick which allows to achieve conditional independence on a large subset of these events.

Lemma 3.2 *Let $\Lambda = (\lambda_{ij})$ be an $N \times N$ matrix such that*
1° $0 \le \lambda_{ij} \le 1$ *for all* i, j
2° $\sum_{i=1}^N \lambda_{ij} = 1$ *for all* j
3° $\lambda_{jj} = 0$ *for all* j.
Then there exists $J \subset \{1, \ldots, N\}$ such that $|J| \ge N/3$ and for every $j \in J$ we have

$$\sum_{i \notin J} \lambda_{ij} \ge 1/3.$$

Proof This lemma is an immediate consequence of the result of K. Ball on suppression of matrices presented and proved in [BT]. By Theorem 1.3 in [BT] applied to Λ, there exists a subset $J \subset \{1, \ldots, N\}$ with $|J| \ge N/3$ such that $\sum_{i \in J} \lambda_{ij} < 2/3$ for $j \in J$, which is just a restatement of the condition in the assertion of the Lemma. □

Now if $\omega \in \Omega \setminus \Theta'_j$, for some $j = 1, \ldots, N$, then, by (3.8) and the definition of $\tilde{D}'_{j,p}$, there exist $x_{i,j} \in F_i \cap B_2^{Nk}$, for all $i \ne j$, with $\sum_{i \ne j} |x_{i,j}|^p = 1$ and $z_j \in \tilde{E}_j \cap B_2^m$ such that

$$\langle \tilde{G}\big(\sum_{i \ne j} x_{i,j}\big), z_j \rangle =: \kappa_j > \kappa.$$

By changing $x_{i,j}$ to $-x_{i,j}$ if necessary, we may assume that $(\tilde{G}x_{i,j}, z_j) \ge 0$ for all $i \ne j$. Thus if $\omega \in \Omega \setminus \Theta'_j$, for all j, then we can consider the matrix Λ defined, for $j = 1, \ldots, N$, by $\lambda_{ij} = \langle \tilde{G}x_{i,j}, z_j \rangle / \kappa_j$ for $i \ne j$ and $\lambda_{jj} = 0$. Let $J \subset \{1, \ldots, N\}$ be the set obtained by Lemma 3.2. Then $|J| \ge N/3$ and for every $j \in J$ we have

$$\langle \tilde{G}\big(\sum_{i \notin J} x_{i.j}\big), z_j \rangle \ge \kappa/3,$$

and so
$$\omega \in \Omega \setminus \left\{ \omega \in \Omega : P_{\tilde{E}_j}(\tilde{D}_{J^c,p}) \subset (\kappa/3) B_2^m \right\};$$
we recall that for a subset $I \subset \{1, \ldots, N\}$, $\tilde{D}_{I,p}$ has been defined in (3.5), and $J^c = \{1, \ldots, N\} \setminus J$.

Let $\mathcal{J}$ be the family of all subsets $J \subset \{1, \ldots, N\}$ with $|J| = \lceil N/3 \rceil =: \ell$. Then the above argument immediately implies that
$$\bigcap_{1 \le j \le N} (\Omega \setminus \Theta'_j) \subset \bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} (\Omega \setminus \Theta'_{j,J^c}), \tag{3.12}$$
where
$$\Theta'_{j,J^c} := \left\{ \omega \in \Omega : P_{\tilde{E}_j}(\tilde{D}_{J^c,p}) \subset (\kappa/3) B_2^m \right\}. \tag{3.13}$$

This definition has a form similar to (3.8) (indeed, $\tilde{D}'_{j,p} = \tilde{D}_{I,p}$ where $I = \{1, \ldots, N\} \setminus \{j\}$; additionally, κ gets replaced by $\kappa/3$). Comparing (3.12) with (3.11) and reintroducing the sets $\Omega \setminus \Theta'_{j,0}$ into our formulae we obtain
$$\Theta^0 \subset \bigcup_{J \in \mathcal{J}} \Theta_J, \tag{3.14}$$
where for $J \in \mathcal{J}$ we set
$$\Theta_J := \bigcap_{j \in J} (\Omega \setminus (\Theta'_{j,J^c} \cap \Theta'_{j,0})). \tag{3.15}$$

Our next objective will be to estimate $\mathbb{P}(\Theta_J)$ for a fixed J. By symmetry, we may restrict our attention to $J = \{1, \ldots, \ell\}$. For $j = 1, \ldots, \ell$, set
$$\mathcal{E}_{j,p} := \Omega \setminus (\Theta'_{j,J^c} \cap \Theta'_{j,0}).$$
Then
$$\Theta_J = \bigcap_{j \in J} \mathcal{E}_{j,p}.$$
We are now in the position to make the key observation of this part of the argument: for a fixed $J \in \mathcal{J}$, the events $\mathcal{E}_{j,p}$, for $j \in J$, are conditionally independent with respect to $\tilde{D}_{J^c,p}$: once $\tilde{D}_{J^c,p}$ is fixed, each $\mathcal{E}_{j,p}$ depends only on the restriction $G_{|F_j}$. In fact, the ensemble $\{\tilde{G}_{|F_j} : j \in J\} \cup \{\tilde{D}_{J^c,p}\}$ is independent since its distinct elements depend on disjoint sets of columns

of $\tilde{G}$, and the columns themselves are independent. This and the symmetry in the indices $j \in J$ implies that

$$\begin{aligned}\mathbb{P}(\Theta_J \mid \tilde{D}_{J^c,p}) &= \mathbb{P}\Big(\bigcap_{j\in J}(\mathcal{E}_{j,p} \mid \tilde{D}_{J^c,p})\Big) \\ &= \prod_{j\in J}\mathbb{P}(\mathcal{E}_{j,p} \mid \tilde{D}_{J^c,p}) = \Big(\mathbb{P}(\mathcal{E}_{1,p} \mid \tilde{D}_{J^c,p})\Big)^{\ell}. \qquad (3.16)\end{aligned}$$

To estimate $\mathbb{P}(\mathcal{E}_{1,p} \mid \tilde{D}_{J^c,p})$ first note that, by the definition of $\mathcal{E}_{1,p}$ this probability is less than or equal to $\mathbb{P}(\Omega \setminus \Theta'_{1,J^c} \mid \tilde{D}_{J^c,p}) + \mathbb{P}(\Omega \setminus \Theta'_{1,0} \mid \tilde{D}_{J^c,p})$. Next, since $\Theta'_{1,0}$ is independent of $\tilde{D}_{J^c,p}$, the second term equals just $1 - \mathbb{P}(\Theta'_{1,0})$. Further, the set $\Theta'_{1,0}$ is the same as in [ST] (where it was denoted by $\Omega'_{1,0}$, see formula (3.7) in that paper), and so

$$\mathbb{P}(\Omega \setminus \Theta'_{1,0} \mid \tilde{D}_{J^c,p}) \le e^{-m/32} + e^{-9m/32}; \qquad (3.17)$$

(see (3.16) in [ST], or use directly Lemma 3.3 from [ST] or Theorem 2.13 from [DS], both of which describe the behaviour of singular numbers of rectangular Gaussian matrices).

For the term involving $\Omega \setminus \Theta'_{1,J^c}$ the probability estimates are much more delicate and will require two auxiliary lemmas. Before we state them, we recall the by now classical concept of functional $M^*(\cdot)$, defined for a set $S \subset \mathbb{R}^d$ by

$$M^*(S) := \int_{S^{d-1}} \sup_{y\in S} \langle x, y\rangle dx, \qquad (3.18)$$

where the integration is performed with respect to the normalized Lebesgue measure on S^{d-1} (this is 1/2 of what geometers call the mean width of S; if S is the unit ball for some norm, $M^*(S)$ is the average of the *dual* norm over S^{d-1}). We then have

Lemma 3.3 *Let d, s be integers with $1 \le d \le s$ and let $A = (a_{ij})$ be a $d \times s$ random matrix with independent $N(0, \sigma^2)$-distributed Gaussian entries. Further, let $a > 0$ and let $S \subset \mathbb{R}^s$ be a symmetric convex body satisfying $S \subset aB_2^s$. Then the random body $AS \subset \mathbb{R}^d$ verifies*

$$\mathbb{E}\left(M^*(AS)\right) = c_s \sigma M^*(S),$$

where $c_s = \sqrt{2}\Gamma(\frac{s+1}{2})/\Gamma(\frac{s}{2}) \le \sqrt{s}$. Moreover, for any $t > 0$,

$$\mathbb{P}\left(M^*(AS) > c_s\sigma M^*(S) + t\right) \le e^{-dt^2/2a^2\sigma^2}.$$

Proof The first assertion is quite standard. We have,

$$\mathbb{E}\left(M^*(AS)\right) = \mathbb{E}\int_{S^{d-1}} \sup_{x\in S} \langle Ax, y\rangle dy = \int_{S^{d-1}} \mathbb{E}\sup_{x\in S} \langle x, A^*y\rangle dy.$$

Since, for any $y \in \mathbb{R}^d$, A^*y is distributed as $\sigma|y|$ times the standard Gaussian vector in $\mathbb{R}^s$, the integrand $\mathbb{E}\sup_{x\in S}\langle x, A^*y\rangle$ does not depend on $y \in S^{d-1}$ and is equal to the appropriate (independent of S) multiple of the spherical mean. The value of the c_s may be obtained, e.g., by calculating the Gaussian average for $S = S^{s-1}$.

For the second assertion, we show first that the function $T \to f(T) := M^*(TS)$ is $a/\sqrt{d}$-Lipschitz with respect to the Hilbert-Schmidt norm $\|\cdot\|_{HS}$. Indeed, directly from the definition (3.18) we have

$$\begin{aligned} f(T_1) - f(T_2) &= \int_{S^{d-1}} \sup_{x\in S} \langle T_1x, y\rangle dy - \int_{S^{d-1}} \sup_{x\in S} \langle T_2x, y\rangle dy \\ &\le \int_{S^{d-1}} \sup_{x\in S} \langle (T_1 - T_2)x, y\rangle dy \\ &\le \int_{S^{d-1}} a|(T_1 - T_2)^*y| dy \\ &\le a\left(\int_{S^{d-1}} |(T_1 - T_2)^*y|^2 dy\right)^{1/2} \\ &= (a/\sqrt{d})\|(T_1 - T_2)^*\|_{HS} = (a/\sqrt{d})\|T_1 - T_2\|_{HS}. \end{aligned}$$

The Gaussian isoperimetric inequality (see e.g., [L], (2.35)) implies now that

$$\mathbb{P}\big(M^*(AS) > \mathbb{E}\left(M^*(AS)\right) + t\big) \le e^{-dt^2/2a^2\sigma^2},$$

which shows that the second assertion of the Lemma follows from the first one. $\square$

The second lemma describes the behaviour of the diameter of a random rank d projection (or the image under a Gaussian map) of a subset of $\mathbb{R}^s$. Let d, s be integers with $1 \le d \le s$ and let $G_{s,d}$ be the Grassmann manifold of d-dimensional subspaces of $\mathbb{R}^s$ endowed with the normalized Haar measure.

Lemma 3.4 *Let $a > 0$ and let $S \subset \mathbb{R}^s$ verify $S \subset aB_2^s$. Then, for any $t > 0$, the set $\left\{H \in G_{s,d} : P_H(S) \subset \left(a\sqrt{d/s} + M^*(S) + t\right) B_2^s\right\}$ has measure*

$\geq 1-\exp(-t^2 s/2a^2+1)$. *Similarly, replacing* P_H *by a* $d\times s$ *Gaussian matrix* A *with independent* $N(0,1/s)$ *entries, we obtain a lower bound on probability of the form* $1-\exp(-t^2 s/2a^2)$.

The phenomenon discussed in the Lemma is quite well known, at least if we do not care about the specific values of numerical constants (which are not essential for our argument) and precise estimates on probabilities. It is sometimes refereed to as the "standard shrinking" of the diameter of a set, and it is implicit, for example, in probabilistic proofs of the Dvoretzky theorem, see [M1], [MS1]. A more explicit statement can be found in [M4], and the present version was proved in [ST].

We now return to the main line of our argument. Observe first that for $1<p\leq 2$ one has

$$M^*\left(\mathrm{conv}_p(F_j\cap B_2^{Nk} : 1\leq j\leq N)\right) = M^*\left(\ell_p^N(\ell_2^k)\right)\leq C_p N^{1/q-1/2}, \quad (3.19)$$

where $q=p/(p-1)$ and $1\leq C_p\leq C\sqrt{q}$, where $C>0$ is an absolute constant. This is likely known, and certainly follows by standard calculations; e.g., by passing to the average of the $\ell_q^N(\ell_2^k)$-norm (dual to the $\ell_p^N(\ell_2^k)$-norm; cf. the comments following (3.18)), expressing it in terms of the Gaussian average and then majorizing the latter via the qth moment, which in turn may be explicitly computed.

The estimate (3.19) has two consequences for the set D_p (defined in (3.3)). Firstly, the Gaussian part of Lemma 3.4 implies that, with our normalization of G, the diameter of D_p is typically comparable to 1. More precisely, consider the exceptional set

$$\Theta^1 := \{\omega : D_p \not\subset 2B_2^n\}. \quad (3.20)$$

Then, as long as $M^*\left(\ell_p^N(\ell_2^k)\right)\leq (n/(4Nk))^{1/2}$, we can apply Lemma 3.4 to the $n\times Nk$ matrix $A=(n/(Nk))^{1/2}G$ and $t=1/2$ to obtain $\mathbb{P}(\Theta^1)\leq\exp(-n/8)$ (note that $a=1$ in this case). On the other hand, by (3.19), the estimate on $M^*\left(\ell_p^N(\ell_2^k)\right)$ is satisfied whenever

$$n\geq 4C_p^2 N^{2/q} k, \quad (3.21)$$

which will be ensured by our final choice of N and the conditions that will be imposed on the dimensions involved.

Secondly, by Lemma 3.3, we have $\mathbb{E}\left(M^*(D_p)\right) \leq C_p\sqrt{k/n}\,N^{1/q}$. (Recall that G is an $n \times Nk$ Gaussian matrix with $\sigma^2 = 1/n$.) Thus, by the second part of the Lemma, our second exceptional set

$$\bar{\Theta}^1 := \{\omega : M^*(D_p) > 2C_p\sqrt{k/n}\,N^{1/q}\} \tag{3.22}$$

satisfies $\mathbb{P}(\bar{\Theta}^1) \leq \exp(-C_p^2 knN^{2/q}/2) \leq \exp(-n/2)$ (remember that $C_p \geq 1$).

Now recall that $Q : \mathbb{R}^n \to \mathbb{R}^m$ is the canonical projection on the first m coordinates. Since $\tilde{D}_p = QD_p$, it follows that for $\omega \notin \Theta^1$ we have

$$\tilde{D}_{J^c,p} \subset \tilde{D}_p \subset 2B_2^m. \tag{3.23}$$

Further, it is a general fact (shown by passing to Gaussian averages) that, for any $S \subset \mathbb{R}^n$, $c_m M^*(QS) \leq c_n M^*(S)$, were c_m and c_n are constants from Lemma 3.3 and $c_n/c_m \leq (2/\sqrt{\pi})\sqrt{n/m}$. Thus, for $\omega \notin \bar{\Theta}^1$ we have

$$M^*(\tilde{D}_{J^c,p}) \leq M^*(\tilde{D}_p) \leq \frac{2}{\sqrt{\pi}}\sqrt{\frac{n}{m}} \cdot 2C_p\sqrt{\frac{k}{n}}\,N^{1/q} = C_p\frac{4}{\sqrt{\pi}}\sqrt{\frac{k}{m}}\,N^{1/q}. \tag{3.24}$$

We now return to our current main task, which is to analyze the set $\Omega \setminus \Theta'_{1,J^c}$. Since we are working with conditional probabilities, we need to introduce another exceptional set which is $\tilde{D}_{J^c,p}$-measurable

$$\Theta' := \{\omega : \tilde{D}_{J^c,p} \not\subset 2B_2^m \text{ or } M^*(\tilde{D}_{J^c,p}) > C_p(4/\sqrt{\pi})\sqrt{k/m}\,N^{1/q}\} \tag{3.25}$$

It follows directly from (3.23) and (3.24) that $\Theta' \subset \Theta^1 \cup \bar{\Theta}^1$. We emphasize that Θ' depends in fact on J, but J is fixed at this stage of the argument. Moreover, the sets Θ' corresponding to different J's are subsets of a small common superset $\Theta^1 \cup \bar{\Theta}^1$, which is additionally independent of Q.

The definition of the set Θ'_{1,J^c} (cf. (3.13)) involves the diameter of a random rank k projection of $\tilde{D}_{J^c,p}$ (note that, by the rotational invariance of the Gaussian measure, $\tilde{E}_1$ is distributed uniformly in $G_{m,k}$, and is independent of $\tilde{D}_{J^c,p}$). Moreover, if $\omega \notin \Theta'$, we control the diameter and M^* of the set $S = \tilde{D}_{J^c,p}$, and so we are exactly in a position to apply Lemma 3.4. Specifically, we use $t = \kappa/6$, $a = 2$ and assume that

$$C_p(4/\sqrt{\pi})\sqrt{k/m}\,N^{1/q} \leq \kappa/12 \tag{3.26}$$

(which implies $a\sqrt{k/m} = 2\sqrt{k/m} \leq \kappa/12$) to obtain

$$\mathbb{P}\left(\Omega \setminus \Theta'_{1,J^c} | \tilde{D}_{J^c,p}\right) \leq e^{-\kappa^2 m/(8\cdot 6^2)+1}. \tag{3.27}$$

For the record, we note that (3.26) implies

$$4C_p^2 N^{2/q} k \leq 4(\kappa/12)^2 m \leq m \leq n,$$

and thus the condition (3.21) that appeared in connection with the measure estimate for Θ^1 is automatically satisfied.

Substituting (3.27) combined with the estimate (3.17) for the measure of $\Omega \setminus \Theta'_{1,0}$ into (3.16) we deduce that, outside of Θ',

$$\begin{aligned}\mathbb{P}(\Theta_J \mid D_{J^c,p}) &\leq \left(e^{-\kappa^2 m/(8\cdot 6^2)+1} + e^{-m/32} + e^{-9m/32}\right)^{\ell} \\ &\leq (2e)^{\ell} e^{-\kappa^2 m\ell/(8\cdot 6^2)}.\end{aligned}$$

Averaging over $\Omega \setminus \Theta'$ (and using $\Theta' \subset \Theta^1 \cup \bar{\Theta}^1$) yields

$$\mathbb{P}\left(\Theta_J \setminus (\Theta^1 \cup \bar{\Theta}^1)\right) \leq \mathbb{P}\left(\Theta_J \setminus \Theta'\right) \leq \mathbb{P}\left(\Theta_J \mid \Omega \setminus \Theta'\right) \leq (2e)^{\ell} e^{-\kappa^2 m\ell/(8\cdot 6^2)}.$$

Since $|\mathcal{J}| = \binom{N}{\ell}$ and $\bigcup_{J\in\mathcal{J}} \Theta_J \supset \Theta^0$ (cf. (3.14)), it follows that

$$\mathbb{P}\left(\Theta^0 \setminus (\Theta^1 \cup \bar{\Theta}^1)\right) \leq \mathbb{P}\left(\bigcup_{J\in\mathcal{J}} \Theta_J \setminus (\Theta^1 \cup \bar{\Theta}^1)\right) \leq \binom{N}{\ell}(2e)^{\ell} e^{-\kappa^2 m\ell/(8\cdot 6^2)}. \tag{3.28}$$

Consequently ,

$$\begin{aligned}\mathbb{P}(\Theta^0) &\leq \mathbb{P}(\Theta^1) + \mathbb{P}(\bar{\Theta}^1) + \mathbb{P}\left(\Theta^0 \setminus (\Theta^1 \cup \bar{\Theta}^1)\right) \\ &\leq e^{-n/8} + e^{-n/2} + \binom{N}{\ell}(2e)^{\ell} e^{-\kappa^2 m\ell/(8\cdot 6^2)}.\end{aligned} \tag{3.29}$$

This ends *Step I* of the proof. To summarize: we have shown that the exceptional set Θ^0 is of exponentially small measure provided (3.26) holds, and that if, additionally, (3.10) is satisfied, then, for $\omega \notin \Theta^0$, the quotient space $\tilde{X}_p$ (obtained from $X_p(\omega)$ via the quotient map Q) contains a well-complemented subspace well-isomorphic to W. To be precise, to arrive at such a conclusion requires optimizing the estimate (3.29) over allowable choices of the parameters N, κ; however, we skip it for the moment since an even more subtle optimization will be performed in *Steps II* and *III*.

Steps II and *III* are very similar as in [ST], Proposition 3.1, so we shall outline the main points only, referring the interested reader to [ST] for details.

Step II. The perturbation argument. Let Q be an *arbitrary* rank m orthogonal projection on $\mathbb{R}^n$. Denote by Θ^Q the set given by formally the same formulae as in (3.11) by the Gaussian operator $\tilde{G} = QG$ for this particular Q. By rotational invariance, all the properties we derived for Θ^0 hold also for Θ^Q. Throughout *Step II*, all references to objects defined in *Step I* will implicitly assume that we are dealing with this particular Q.

Consider the exceptional set Θ^1 defined in (3.20), and observe that if $\omega \notin \Theta^1$, then

$$D'_{j,p} \subset D_p \subset 2B_2^n, \tag{3.30}$$

for every $j = 1, \ldots, N$. This is an analogue of (3.26) of [ST] and the basis for all the estimates that follow.

Let $\omega \notin \Theta^1 \cup \Theta^Q$ and let Q' be any rank m orthogonal projection such that $\|Q - Q'\| \leq \delta$, where $\|\cdot\|$ is the operator norm with respect to the Euclidean norm $|\cdot|$ and $\delta > 0$ will be specified later. Then, for some j, conditions just slightly weaker than those in (3.8) and (3.9) hold with Q replaced by Q'. Namely, there exists $1 \leq j \leq N$ such that, firstly, if $\delta \leq (1/8)\sqrt{m/n}$ then Q' satisfies inclusions analogous to (3.9) with constants $1/2$ and 2 replaced by $1/4$ and $9/4$, respectively (cf. (3.28) of [ST]); and, secondly, if $\delta_1 := 4\delta\sqrt{n/m} \leq \kappa/4$ then Q' satisfies inclusions analogous to (3.8) with κ replaced by 2κ. (The former statement is exactly the same as in [ST], and the proof of the latter uses the above inclusion (3.30) instead of (2.26) of [ST].)

Finally, set $\delta := 1/(8\sqrt{n})$ (as in [ST]); then the condition $\delta \leq (1/8)\sqrt{m/n}$ is trivially satisfied, while the condition $\delta_1 \leq \kappa/4$ follows from (3.26). So we can now apply the previous arguments and conclude *Step II*: if $\omega \notin \Theta^1 \cup \Theta^Q$, $\|Q - Q'\| \leq \delta$ and

$$2\kappa \leq \frac{1}{\sqrt{k}} \cdot \frac{1}{4}\sqrt{\frac{m}{n}}, \tag{3.31}$$

then the quotient of X_p corresponding to Q' contains a $2^{1/q}$-complemented subspace $2^{1/q}$-isomorphic to W, namely $Q'E_j$. We note that (3.31) is just slightly stronger than (3.10), and as easy to satisfy.

Step III. The discretization: a δ-net argument. Let $\mathcal{Q}$ be a δ-net in the set of rank m orthogonal projections on $\mathbb{R}^n$ endowed with the distance given by the operator norm. Recall that such a net can be taken with cardinality $|\mathcal{Q}| \leq (C_2/\delta)^{m(n-m)}$, where C_2 is a universal constant (see [ST], or directly [S2]). For our choice of $\delta = 1/(8\sqrt{n})$, this does not exceed $e^{mn\log n}$, at least

for sufficiently large n. As in (3.28)-(3.29), this implies the measure estimate for our final exceptional set

$$\begin{aligned} \mathbb{P}\left(\Theta^1 \cup \bigcup_{Q\in\mathcal{Q}} \Theta^Q\right) &\leq \mathbb{P}\left(\Theta^1 \cup \bar{\Theta}^1 \cup \bigcup_{Q\in\mathcal{Q}} (\Theta^Q \setminus (\Theta^1 \cup \bar{\Theta}^1))\right) \\ &\leq e^{-n/8} + e^{-n/2} + e^{mn\log n}\binom{N}{\ell}(2e)^\ell e^{-\kappa^2 m\ell/32} \end{aligned} \tag{3.32}$$

The first two terms are negligible. Recall that $\ell = \lceil N/3 \rceil \geq N/3$, and so the last term in (3.32) is less than or equal to $e^{mn\log n - \kappa^2 mN/128}$.

In conclusion, if k, κ, m, n and N satisfy

$$C\sqrt{q}\,(4/\sqrt{\pi})\sqrt{k/m}\,N^{1/q} \leq \kappa/12, \qquad 256\,mn\log n \leq \kappa^2 mN, \tag{3.33}$$

where $C > 0$ is the absolute constant related to C_p (see (3.26) and (3.19)), then the set $\Omega \setminus (\Theta^1 \cup \bigcup_{Q\in\mathcal{Q}} \Theta^Q)$ has positive measure (in fact, very close to 1 for large n). If, additionally, (3.31) is satisfied, then any ω from this set induces an n dimensional space X_p whose *all* m-dimensional quotients contain a $2^{1/q}$-isomorphic and $2^{1/q}$-complemented copy of W (and similarly with $1+\varepsilon$ in place of $2^{1/q}$ if (3.31) holds with an additional ε factor on the right hand side). Then the assertion of Theorem 2.1 holds for that particular value of m.

It remains to ensure that conditions (3.33) and (3.31) are consistent and to discuss the resulting restrictions on the dimensions. It is most convenient to let $\kappa := (1/8)\sqrt{m/(nk)}$ so that (3.31) holds. Then the conditions in (3.33) lead to

$$k \leq c' \min\left\{\frac{m}{\sqrt{qn}N^{1/q}}, \frac{mN}{n^2\log n}\right\}, \tag{3.34}$$

where $c' \in (0,1)$ is a universal constant. Optimizing over N leads to

$$k \leq \frac{c_1 m}{q^{1/2}\ n^{(4+q)/(2+2q)}\ (\log n)^{1/(1+q)}}.$$

which, for $m = m_0$, is just a rephrasing of the hypothesis on $\dim V = \dim W$ from Theorem 2.1, and holds in the entire range $m_0 \leq m \leq n$ if it holds for m_0. It follows that, under our hypothesis, the above construction can be implemented for each m verifying $m_0 \leq m \leq n$. Moreover, since the estimates on the probabilities of the exceptional sets corresponding to different values

of m are exponential in $-n$ (as shown above), the sum of the probabilities involved is small. Consequently, the construction can be implemented *simultaneously* for all such m with the resulting space satisfying the full assertion of Theorem 2.1 with probability close to 1.

Finally, we point out that, as it was already alluded to earlier at some crucial points of the argument, the $1+\varepsilon$-version of the statement will follow once our parameters satisfy (3.33) and the condition analogous to (3.31), with an extra ε on the right hand side. With the choice of $\kappa := (\varepsilon/8)\sqrt{m/(nk)}$, this leads to a version of (3.34), which – after optimizing over N – gives the same bound for k as above, but with the constant c_1 depending on ε rather than being universal. The rest of the argument is the same. $\square$

4 The global saturation

Proof of Theorem 2.2 Let W be a k-dimensional normed space. Identify W with $\mathbb{R}^k$ in such a way that $(1/\sqrt{k})B_2^k \subset B_W \subset B_2^k$.

We use an analogous notation for convex bodies as in the the proof of Theorem 2.1 (but without the subscript p). In particular, we set $Z = \ell_1^N(W)$ and we recall that $G = G(\omega)$ denotes a $n \times Nk$ random matrix with independent $N(0, 1/n)$-distributed Gaussian entries. We let

$$K = B_{X(\omega)} := G(\omega)(B_Z) \subset \mathbb{R}^n.$$

Recall that for $j = 1, \ldots, N$, F_j is the k-dimensional coordinate subspace of $\mathbb{R}^{Nk}$ corresponding to the jth consecutive copy of W in Z; $E_j := G(F_j)$, $K_j := G(F_j \cap B_Z)$ and $K'_j := G\left(\operatorname{span}\left[F_i : i \neq j\right] \cap B_Z\right) = \operatorname{conv}\left(K_i : i \neq j\right)$; next, $D_j := G(F_j \cap B_2^{Nk})$ and $D'_j := \operatorname{conv}\left(D_j : i \neq j\right)$. (The notation D_j has been already used in the proof of Theorem 2.1, and the "p-convex" analogoue of D'_j, namely $D'_{j,p}$, was defined in (3.4).)

The general structure of the argument is the same as in Theorem 2.1: the proof consists of three steps dealing respectively with analysis of a single rotation, perturbation of a given rotation and discretization (for a smoother narrative, here and in what follows we refer to elements of $O(n)$ – even those whose determinant is not 1 – as rotations). We will refer extensively to arguments in Section 3 and in [ST]. As in Section 3, we shall occasionally assume, as we may, that n is large.

Step I. Probability estimates for a fixed rotation. For the time being we fix $u : \mathbb{R}^n \to \mathbb{R}^n$ with $u \in O(n)$. We shall show that, outside of an exceptional set of ω's of a small measure, there is a section of $K + u(K)$ which is 3-isomorphic to B_W and 3-complemented (or, more precisely, that the identity on W 3-factors through the space $(\mathbb{R}^n, K + u(K))$.

We shall adopt the following description of the body $K + u(K)$. Let $B_Z \oplus_\infty B_Z$ be the unit ball of $Z \oplus_\infty Z$ (i.e., $\mathbb{R}^{Nk} \oplus \mathbb{R}^{Nk}$ with the ℓ_∞-norm on the direct sum). Next, consider the Gaussian operator $G \oplus G : \mathbb{R}^{Nk} \oplus \mathbb{R}^{Nk} \to \mathbb{R}^n \oplus \mathbb{R}^n$, acting in the canonical way on the coordinates. Further, define $[\mathrm{Id}, u] : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n$ by $[\mathrm{Id}, u](x_1, x_2) = x_1 + ux_2$, for $(x_1, x_2) \in \mathbb{R}^n \oplus \mathbb{R}^n$. Clearly, we have $K + u(K) = [\mathrm{Id}, u](G \oplus G)(B_Z \oplus_\infty B_Z)$. Instead of $[\mathrm{Id}, u]$ we can equally well use $[u_1, u_2]$, where $u_1, u_2 \in O(n)$ are two rotations.

The difference between this setup and the scheme of [ST] is that in the latter one considers $QG''(B_Z \oplus_1 B_Z)$, where G'' is a $2n \times 2Nk$ Gaussian matrix and Q a rank n orthogonal projection on $\mathbb{R}^{2n}$. Both schemes yield quotients of random quotients of $Z \oplus Z$, with $G \oplus G$ or G'' being the random part and $[u_1, u_2]$ or Q the nonrandom part. For the latter one may as well "rescale" the dimensions and consider $Q'G(B_Z)$, where Q' is a (nonrandom) rank $\lfloor n/2 \rfloor$ projection. The setting in Section 3 is identical, except that we consider $B_{Z_p} \oplus_p B_{Z_p}$ instead of $B_Z \oplus_1 B_Z$.

To define exceptional sets we identify conditions similar to those in Section 3 (or in Section 3 of [ST]). Recall that for $E \subset \mathbb{R}^n$, by P_E we denote the orthogonal projection onto E. Now, for $j \in \{1, \dots, N\}$, and $0 < \kappa < 1$ fixed, to be specified later, we consider the set

$$\Xi'_j := \left\{ \omega \in \Omega : P_{E_j}(D'_j + u(D'_j)) \subset \kappa B_2^n \right\}. \tag{4.1}$$

These sets are analogous to Θ'_j in (3.8), and they will replace these latter sets in all subsequent definitions. A similar proof as for (3.27) in Section 3, or (3.23) of [ST], shows that

$$\mathbb{P}(\Xi'_j) \geq 1 - \exp(-c_1 \kappa^2 n), \tag{4.2}$$

as long as

$$\kappa \geq C' \sqrt{\max\{k, \log N\}/n}, \tag{4.3}$$

for appropriate numerical constants $c_1 > 0$ and $C' \geq 1$. The argument is again based on Lemma 3.4: since E_j is independent of D'_j, we may as well consider it fixed, and then we are exactly in the setting of the Gaussian part

of the Lemma. We just need to majorize $M^*(B_Z)$ (or, more precisely, just of the unit ball of $\ell_1^{N-1}(\ell_2^k)$ since the ℓ_2^k-factor corresponding to F_j does not enter into D'_j), which is $O(\sqrt{\max\{k, \log N\}/n})$ by reasons similar to – but simpler than – those that led to (3.20) of [ST] (the calculations sketched in the paragraph containing (3.19) give a slightly larger majorant, which would also suffice for our purposes).

Next, for $j = 1, \ldots, N$ we let

$$\Xi'_{j,0} := \{\omega \in \Omega : (1/2)\,(B_2^n \cap E_j) \subset D_j\}\,. \tag{4.4}$$

Since the condition in (4.4) involves only one of the two inclusions appearing in (3.9), the same argument that led to (3.17) (see also (3.16) of [ST]) gives

$$\mathbb{P}(\Xi'_{j,0}) \geq 1 - \exp(-n/32). \tag{4.5}$$

While in Theorem 2.1 and in [ST], properties analogous to those implicit in the definitions of the sets Ξ'_j, $\Xi'_{j,0}$ were sufficient to ensure that the quotient $Q(K)$ contained a well-complemented subspace well-isomorphic to W, this is not the case in the present context and we need to introduce additional invariants.

Fix $\alpha_0 > 0$ to be specified later (it will be of the order of $1/k$). Let $\alpha := \mathrm{tr}\,(\mathrm{Id} - u)/n$, and assume without loss of generality that $0 \leq \alpha \leq 1$ (replacing, if necessary, u by $-u$). The proof now splits into two cases depending on whether $\alpha \geq \alpha_0$ or $\alpha < \alpha_0$. To clarify the structure of the argument let us mention that, among the sets Ξ'_j and $\Xi'_{j,0}$ defined above, *Case* 1° will use only the former ones, while *Case* 2° will involve both.

Case 1°: Let $\alpha \geq \alpha_0$.

Lemma 4.1 *Let A be an $n \times k$ random matrix with independent $N(0, 1/n)$-distributed Gaussian entries. Let $u \in O(n)$ with $\mathrm{tr}\, u \geq 0$ and set $\alpha = \mathrm{tr}\,(\mathrm{Id} - u)/n$ ($\in [0, 1]$). Then, with probability greater than or equal to $1 - \exp(-c\alpha n + c^{-1} k \log(2/\alpha))$, the following holds for all $\xi, \zeta \in \mathbb{R}^k$*

$$|A\xi + uA\zeta| \geq c\alpha^{1/2} \left(|\xi|^2 + |\zeta|^2\right)^{1/2} \geq (c/2)\alpha^{1/2} \left(|A\xi|^2 + |uA\zeta|^2\right)^{1/2}, \tag{4.6}$$

where $c > 0$ is a universal constant.

We postpone the proof of the lemma until the end of the section and continue the main line of the argument. For $j = 1, \ldots, N$ we let

$$H_j = E_j + u(E_j).$$

We shall now use Lemma 4.1 for the $n \times k$ matrix $A = A_j$ formed by the k columns of the matrix G that span E_j. Denoting by $\Xi_{j,0}$ the subset of Ω on which the inequalities (4.6) holds, we have

$$\begin{aligned} \mathbb{P}(\Xi_{j,0}) &\geq 1 - \exp(-c\alpha n + c^{-1} k \log(1/\alpha)) \\ &\geq 1 - \exp(-c\alpha_0 n + c^{-1} k \log(1/\alpha_0)). \end{aligned} \tag{4.7}$$

Consider the following auxiliary set, closely related to $\Xi_{j,0}$,

$$\Delta_j := \left\{ \omega \in \Omega : c\alpha^{1/2}(B_2^n \cap H_j) \subset D_j + u(D_j) \text{ and } \dim H_j = 2k \right\}. \tag{4.8}$$

An elementary argument shows that the conditions in (4.8) are equivalent to "$|A\xi + uA\zeta| \geq c\alpha^{1/2} \max\{|\xi|, |\zeta|\}$ for all $\xi, \zeta \in \mathbb{R}^k$." Since this is weaker than the first inequality in (4.6), it follows that $\Xi_{j,0} \subset \Delta_j$.

Our next objective is to show that on $\Xi_{j,0}$

$$|P_{H_j} z| \leq (2/c)\alpha^{-1/2} \Big(|P_{E_j} z|^2 + |P_{u(E_j)} z|^2 \Big)^{1/2}, \tag{4.9}$$

for every $z \in \mathbb{R}^n$.

Note that since E_j and $u(E_j)$ are both subspaces of H_j, it is sufficient to assume that $z \in H_j$. Consider the operator $T : H_j \to E_j \oplus_2 u(E_j)$ given by $T(z) = (P_{E_j} z, P_{u(E_j)} z)$ for $z \in H_j$. Then the inequality (4.9) is equivalent to $\|T^{-1}\| \leq (2/c)\alpha^{-1/2}$. On the other hand, the adjoint operator $T^* : E_j \oplus_2 u(E_j) \to H_j$ is given by $T^*(x, y) = x + y$ for $x \in E_j$ and $y \in u(E_j)$. Comparing the first and the third terms of (4.6) yields $\|T^{-1}\| = \|(T^*)^{-1}\| \leq (2/c)\alpha^{-1/2}$, as required.

Finally, consider another good set

$$\Xi_j'' := \left\{ \omega \in \Omega : P_{u(E_j)}(D_j' + u(D_j')) \subset \kappa B_2^n \right\}. \tag{4.10}$$

Note that since u is orthogonal, we clearly have $P_{u(E_j)} = u P_{E_j} u^*$ (this will be used more than once). Comparing (4.10) with the definition of Ξ_j' (see (4.1)), we deduce from (4.2) that

$$\mathbb{P}(\Xi_j'') = \mathbb{P}(\Xi_j') \geq 1 - \exp(-c_1 \kappa^2 n) \tag{4.11}$$

We are now ready to complete the analysis specific to *Case* 1°. Let $\omega \in \Xi_{j,0} \cap \Xi'_j \cap \Xi''_j$. Then, combining (4.9) with the definitions of Ξ'_j and Ξ''_j – i.e., with (4.1) and (4.10) – we see that, for all $z \in D'_j + u(D'_j)$,

$$|P_{H_j} z| \le (2/c)\alpha^{-1/2}\Big(|P_{E_j} z|^2 + |P_{u(E_j)} z|^2\Big)^{1/2} \le (2\sqrt{2}/c)\alpha^{-1/2}\kappa$$

or, equivalently,

$$P_{H_j}(D'_j + u(D'_j)) \subset (2\sqrt{2}/c)\alpha^{-1/2}\kappa B_2^n \tag{4.12}$$

As in the previous proofs we will impose a condition on κ, namely

$$(2\sqrt{2}/c)\alpha_0^{-1/2}\kappa \le c\alpha_0^{1/2}/\sqrt{k}. \tag{4.13}$$

Combining this inequality with (4.12) and (4.8), and recalling that $\Xi_{j,0} \subset \Delta_j$ and that $\alpha_0 \le \alpha$, we are led to

$$P_{H_j}(D'_j + u(D'_j)) \subset 1/\sqrt{k}\,(D_j + u(D_j))\,.$$

Finally, recalling the inclusions between the K- and the D-sets, we obtain

$$P_{H_j}(K'_j + u(K'_j)) \subset K_j + u(K_j.$$

Consequently, similarly as in the previous proofs (cf. (3.6), or (3.3) of [ST]),

$$P_{H_j}(K + u(K)) \subset \mathrm{conv}\,\big(K_j + u(K_j), P_{H_j}(K'_j + u(K'_j))\big) \subset K_j + u(K_j).$$

This means that $K_j + u(K_j)$ is a 1-complemented section of $K + u(K)$. On the other hand, let us note that, again by (4.8), $\dim H_j = 2k$, which implies that $K_j + u(K_j)$ (thought of as a normed space) is isometric to $B_W \oplus_\infty B_W$, thus showing that $H_j \cap (K + u(K))$ is isometric to $B_W \oplus_\infty B_W$ as well.

We recall that the above conclusion was arrived at under the hypothesis $\omega \in \Xi_{j,0} \cap \Xi'_j \cap \Xi''_j$. As $j \in \{1, \ldots, N\}$ was arbitrary, we deduce that under the hypothesis of *Case* 1° and the additional assumptions (4.3) and (4.13), the set $K + u(K)$ admits a 1-complemented section isometric to B_W provided that $\omega \in \bigcup_{j=1}^N (\Xi_{j,0} \cap \Xi'_j \cap \Xi''_j)$.

Case 2°: Let $\alpha < \alpha_0$.

In this case the operator u is close to the identity operator. In particular, since $\alpha = \mathrm{tr}\,(\mathrm{Id} - u)/n$, we see that the norm

$$\|\,\mathrm{Id} - u\|_{HS} = (\mathrm{tr}\,(\mathrm{Id} - u)(\mathrm{Id} - u^*))^{1/2} = (2(n - \mathrm{tr}\ u))^{1/2} = (2n\alpha)^{1/2}$$

is relatively small. To exploit this property we will need another lemma.

Lemma 4.2 *Let A be an $n \times k$ random matrix with independent $N(0,1/n)$-distributed Gaussian entries. Let T be an $n \times n$ matrix, set $a := \|T\|_{HS}$ and let $\gamma > 0$. Then, on a set of probability larger than or equal to $1 - \exp\left(-\gamma^2 n/(2\|T\|^2) + 2k\right)$, the following holds for all $\xi = (\xi_i) \in \mathbb{R}^k$*

$$|TA\xi| \leq 2\left(\frac{a}{\sqrt{n}} + \gamma\right)|\xi|. \tag{4.14}$$

Again, we postpone the proof of the Lemma and continue our argument. Fix $\gamma > 0$, to be specified later. For $j = 1, \ldots, N$, let

$$\Xi''_{j,0} := \left\{\omega \in \Omega : (\mathrm{Id} - u)D_j \subset 2(\sqrt{2\alpha} + \gamma)B_2^n\right\}. \tag{4.15}$$

As was the case with Lemma 4.1, we shall apply the Lemma to the $n \times k$ matrix $A = A_j$ formed by the k columns of the matrix G that span E_j. We will also use $T = \mathrm{Id} - u$, so that $\|T\| \leq 2$. Since, in that case, $a/\sqrt{n} = \sqrt{2\alpha}$, the inclusion from (4.15) is equivalent to the inequality (4.14) and thus

$$\mathbb{P}(\Xi''_{j,0}) \geq 1 - \exp(-\gamma^2 n/8 + 2k). \tag{4.16}$$

The latter expression will be later made very close to 1 by an appropriate choice of parameters.

Next we shall show that if $j \in \{1, \ldots, N\}$ and $\omega \in \Xi'_j \cap \Xi'_{j,0} \cap \Xi''_{j,0}$, then

$$K_j \subset P_{E_j}(K + u(K)) \subset 3K_j. \tag{4.17}$$

Clearly, this will imply that the section of $K + u(K)$ by E_j is 3-isomorphic to K_j, which in turn is isometric to B_W; and additionally, that it is 3-complemented. Consequently, under the hypothesis of *Case* $2°$, the assertion of *Step I* will be shown to hold on the set $\bigcup_{j=1}^N (\Xi'_{j,0} \cap \Xi''_{j,0} \cap \Xi'_j)$.

To show (4.17), we first point out that if $B \subset \mathbb{R}^n$ is *any* symmetric convex body, then $B + u(B) \subset 2B + (\mathrm{Id} - u)(B)$. We then argue as follows

$$\begin{aligned} K + u(K) &\subset K_j + K'_j + u(K_j) + u(K'_j) \\ &\subset K_j + D'_j + u(K_j) + u(D'_j) \\ &\subset 2K_j + (\mathrm{Id} - u)K_j + \left(D'_j + u(D'_j)\right) \\ &\subset 2K_j + 2(\sqrt{2\alpha} + \gamma)B_2^n + \left(D'_j + u(D'_j)\right), \end{aligned}$$

where the last inclusion is a consequence of (4.15). Accordingly

$$\begin{aligned} P_{E_j}\left(K+u(K)\right) &\subset 2K_j + P_{E_j}\left(D'_j + u(D'_j)\right) + 2(\sqrt{2\alpha}+\gamma)B_2^n \cap E_j \\ &\subset 2K_j + (\kappa + 2\sqrt{2\alpha}+2\gamma)(B_2^n \cap E_j), \end{aligned}$$

with the last inclusion following from the definition (4.1) of set Ξ'_j. By the definition (4.4) of $\Xi'_{j,0}$, the second term on the right is contained in $2(\kappa + 2\sqrt{2\alpha}+2\gamma)D_j$. Since $\alpha < \alpha_0$, it follows that whenever

$$2(\kappa + 2\sqrt{2\alpha_0}+2\gamma) \le 1/\sqrt{k}, \tag{4.18}$$

then

$$P_{E_j}\left(K+u(K)\right) \subset 2K_j + (1/\sqrt{k})D_j \subset 3K_j.$$

We thus obtained the right hand side inclusion in (4.17); the left hand side inclusion is trivial. This ends the analysis specific to *Case* 2°.

Now is the time to choose α_0 and γ to satisfy our restrictions while yielding the optimal concentration in *both* cases under consideration. The conditions (4.3), (4.13) and (4.18) can be summarized as $C'\sqrt{\max\{k, \log N\}/n} \le \kappa \le c'\alpha_0/\sqrt{k}$ and $\max\left\{\sqrt{\alpha_0}, \gamma\right\} \le c'/\sqrt{k}$, for appropriate numerical constants $c' > 0$ and $C' \ge 1$. We choose α_0, γ and κ so that

$$\kappa^{1/3} = \sqrt{\alpha_0} = \gamma = c'/\sqrt{k} \tag{4.19}$$

This choice takes care of all the restrictions except for the lower bound on κ, which can be now rephrased as

$$k \le c\min\{n^{1/4}, (n/\log N)^{1/3}\}, \tag{4.20}$$

for an appropriate numerical constant $c > 0$.

We shall now analyze the estimates on the probabilities of the good sets contained in (4.16), (4.7) and (4.11). If k^2/n is sufficiently small, a condition which is weaker than (4.20), then the term $2k$ in the exponent in (4.16) is of smaller order than the first term, and so it does not affect the form of the estimate. The situation is slightly more complicated in the case of (4.7): to absorb the second term in the exponent we need to know that $k\log(1/\alpha_0)$ is sufficiently smaller than $\alpha_0 n$; , given that $\alpha_0 = O(1/k)$ (cf. (4.19)), this is equivalent to

$$k \le c'' \sqrt{\frac{n}{1+\log n}}$$

for an appropriate numerical constant $c'' > 0$. Again, this is a condition weaker than (4.20), at least for sufficiently large n. The probability estimates in question are thus, respectively, of the form $1 - \exp(-c_3\gamma^2 n)$, $1 - \exp(-c_2\alpha_0 n)$ and $1 - \exp(-c_1\kappa^2 n)$, for appropriate universal constants $c_1, c_2, c_3 > 0$. Substituting the values for α_0, γ and κ defined by (4.19) we get, under the hypothesis (4.20), the following minoration

$$\min\{\mathbb{P}(\Xi'_j), \mathbb{P}(\Xi''_j), \mathbb{P}(\Xi_{j,0}), \mathbb{P}(\Xi''_{j,0}), \mathbb{P}(\Xi'_{j,0})\} \geq 1 - \exp(-c_0 n/k^3), \quad (4.21)$$

again for an appropriate numerical constant $c_0 > 0$. We point out that the argument above treated just the first four terms under the minimum; for $\mathbb{P}(\Xi'_{j,0})$ we have the stronger estimate (4.5), which does not require any additional assumptions.

We are now ready to conclude *Step I.* Consider the exceptional set defined by one of two different formulae, depending on whether we are in *Case* 1° or *Case* 2°. In *Case* 1° we set

$$\Xi^0 := \Omega \setminus \bigcup_{j=1}^{N} (\Xi_{j,0} \cap \Xi'_j \cap \Xi''_j)$$

(see (4.6) and the paragraph following it, (4.1), (4.10) for the definitions). In *Case* 2° we let

$$\Xi^0 := \Omega \setminus \bigcup_{j=1}^{N} (\Xi'_{j,0} \cap \Xi''_{j,0} \cap \Xi'_j),$$

(see (4.4), (4.15) and (4.1) for the definitions). The argument above shows that for $\omega \notin \Xi^0$ there is a section of $K + u(K)$ 3-isomorphic to B_W and 3-complemented.

It follows readily from what we have shown up to now that the sets Ξ^0 are exponentially small. For example, by (4.21),

$$\mathbb{P}\big(\Omega \setminus (\Xi_{j,0} \cap \Xi'_j \cap \Xi''_j)\big) \leq 3\exp(-c_0 n/k^3) \quad (4.22)$$

for any $j \in \{1, \dots, N\}$, and identical estimates hold for exceptional sets relevant to *Case* 2°. However, to finalize *Step I* we need to majorize the probability of Ξ^0 much more efficiently. To this end we argue in the same way as in Section 3 of [ST]. We could also follow the argument from Section 3 above, but in the present situation, when we are dealing with the convex hulls of sets – such as K_i or D_j – rather then the p-convex hulls of

the same sets, with $p > 1$, the latter option would only add unnecessary complications. However, for reader's convenience, we will also include a few comments pertaining to the proof of Theorem 2.1.

We first employ the "decoupling" procedure based on Lemma 3.2 in [ST] (which is a special case of Lemma 3.2 above for a "0-1" matrix A). More precisely, we do need and do have estimates on conditional probabilities which are obtained in essentially the same way as there (and are also parallel to the estimates for Θ^0 earlier in this paper). Essential use is also made of the exceptional set

$$\Omega^1 := \{\omega : D \not\subset 2B_2^n\}$$

(defined in (3.17) of [ST] and analogous to Θ^1 in Section 3) and the precise statements involve Ω^1 and sets related to it. Again, the key point is that the linear subspace E_j (resp. $E_j + u(E_j)$) and the sets with which it is being intersected (or which are projected onto it) depend on disjoint blocks of columns of G and hence are independent. The decoupling procedure and the estimate from (4.21) lead to

$$\mathbb{P}(\Xi^0) \leq Ne^{-9n/32} + \binom{N}{\ell}\left(3e^{-c_0 n/k^3}\right)^{\ell} \leq Ne^{-9n/32} + e^{-c_4 N/k^3}, \qquad (4.23)$$

where $\ell = \lceil N/3 \rceil$ (cf. (4.22)). This is almost identical to (3.25) of [ST] (and analogous to (3.29) above). Let us emphasize that the set Ω^1, responsible for the first term of the estimate, is independent on u, and therefore, when (4.23) is combined with the δ-net argument in *Step III* below, only the second term will have to be multiplied by the cardinality of the net.

Step II. Stability under small perturbations of the rotation u. We will now prove that there exists (a not too small) $\delta > 0$ such that if $u \in O(n)$ and $\omega \notin \Xi^0$ (where Ξ^0 is defined starting with this particular u) and if $u' \in O(n)$ with $\|u - u'\| \leq \delta$, then u' and ω satisfy essentially the same conditions as those defining Ξ^0. As in [ST] (and analogously as in Section 3 above), this will be shown under an additional assumption, namely that $\omega \notin \Omega^1$ (the definition of Ω^1 was recalled above). It will then follow that, for any u' as above, the random body K corresponding to any $\omega \notin \Omega^1 \cup \Xi^0$ will have the property that $K + u'(K)$ has a section that is 3-isomorphic to B_W and 3-complemented provided the parameters involved in the construction satisfy conditions differing from those of *Step I* (which, we recall, were ultimately reduced just to (4.20)) only by values of the numerical constants.

We start by pointing out that the condition (4.4) does not involve u and so it is trivially stable. Next, we consider (4.1) which, while non-trivial, is easy to handle. We have

$$u'(D'_j) \subset u(D'_j) + 2\delta B_2^n$$

(because $\omega \notin \Omega^1$) and so if $\delta \leq \kappa/2$, we get (4.1) for u' in place of u, at the cost of replacing κ by 2κ on the right hand side of the inequality.

The condition (4.15) is also simple: if $\omega \notin \Omega^1$ and $\delta \leq \gamma$, and if (4.15) is satisfied for u, then it is clearly satisfied for u' with the factor 2 on the right hand side replaced by 3. (Note that this argument works for a general u, even though the condition (4.15) enters the proof only in *Case* $2°$.)

Next we assume that we are in *Case* $1°$ and discuss the stability of $\Xi_{j,0}$, defined by inequality (4.6) (where the matrix $A = A_j$ has been described in the paragraph following Lemma 4.1). We clearly have

$$\begin{aligned} |A_j\xi + u'A_j\zeta| &\geq |A_j\xi + uA_j\zeta| - \|u - u'\|\,|A_j\zeta| \\ &\geq c\alpha^{1/2}\left(|\xi|^2 + |\zeta|^2\right)^{1/2} - 2\delta|\zeta|. \end{aligned}$$

So if $\delta \leq c\alpha^{1/2}/4$, we get a version of the first inequality in (4.6) with u' in place of u and c on the right hand side replaced by $c/2$. The second inequality follows similarly. Since (given that we are in *Case* $1°$) $\alpha \geq \alpha_0$, we see that the condition on δ is satistfied when $\delta \leq c\alpha_0^{1/2}/4$.

It remains to check the stability of (4.10). Set $R = u' - u$, then $\|R\| \leq \delta$ and, using $P_{u'(E_j)} = u'P_{E_j}u'^*$, we obtain

$$\begin{aligned} P_{u'(E_j)}(D'_j + u'(D'_j)) &= u'P_{E_j}(u'^*D'_j + D'_j) \\ &= (u+R)P_{E_j}((u^* + R^*)D'_j + D'_j) \\ &\subset (u+R)P_{E_j}\left((u^*D'_j + D'_j) + 2\delta B_2^n\right) \\ &\subset uP_{E_j}(u^*D'_j + D'_j) + 2\delta uP_{E_j}B_2^n + RP_{E_j}(4B_2^n) \\ &\subset uP_{E_j}(u^*D'_j + D'_j) + 6\delta B_2^n. \end{aligned}$$

Since $uP_{E_j}u^* = P_{u(E_j)}$, insisting that $\delta \leq \kappa/6$ will guarantee that u' satifies the inclusion from (4.10) with κ replaced by 2κ.

Finally, let us remark that the distinction between *Cases* $1°$ and $2°$ is likewise essentially stable under small perturbations in u: the parameter α is 1-Lipschitz with respect to the operator norm and so if δ is less than $1/2$ of the threshold value $\alpha_0 = c'^2/k$, then the inequalities defining *Case* $1°$ and

2° will have to be modified at most by factor 2 when passing from u to u' (or vice versa).

Comparing the obtained conditions on δ we see that the most restrictive is $\delta \leq \kappa/6 = c''' k^{-3/2}$. Since, by (4.20) (and, ultimately, by the hypothesis of the Theorem), k is at most of the order of $n^{1/4}$, the appropriate choice of $\delta = O(n^{-3/8})$, will cover the entire range of possible values of k. This supplies the value of δ that needs to be used in the discretization (a δ-net argument) to be implemented in *Step III* below.

Step III. A discretization argument. The procedure is fully parallel to that of Section 3: we introduce a δ-net of $O(n)$, say $\mathcal{U}$, and then combine the exceptional sets corresponding to the elements of $\mathcal{U}$. For the argument to work, it will be sufficient that the cardinality of $\mathcal{U}$ multiplied by the probability of the exceptional set corresponding to a particular rotation u (i.e., the second term at the right hand side of (4.23)) is small. As is well known (see, e.g., [S1], [S2]), $O(n)$ admits, for any $\delta > 0$, a δ-net (in the operator norm) of cardinality not exceeding $(C/\delta)^{\dim O(n)}$, where C is a universal constant. Our choice of $\delta = O(1/n^\beta)$ (where $\beta = 3/8$, see the last paragraph of *Step II*) leads to the estimate

$$\log |\mathcal{U}| \leq O(\beta n^2 (1 + \log n)).$$

For the probability of combined exceptional sets to be small it will thus suffice that, for an appropriately chosen $c_5 > 0$,

$$\beta n^2 (1 + \log n) \leq c_5 N/k^3$$

(cf. (4.23)). Since, as in the argument at the end of *Step II*, we may assume that k is at most of the order of $n^{1/4}$, the condition above may be satisfied in the entire range of possible values of k with $N = O(n^{11/4}(1 + \log n))$. Since such a choice implies that ue have then $\log N = O(\log n)$, the restrictions given by (4.20)) reduce, at least for large n, to $k \leq cn^{1/4}$ – exactly the hypothesis of the Theorem. $\square$

To complete the proof of Theorem 2.2 it remains to prove Lemmas 4.1 and 4.2. The arguments are fairly straightforward applications of the Gaussian isoperimetric inequality, or Gaussian concentration, again in the form given, e.g., in [L], formula (2.35).

Proof of Lemma 4.1 Fix $\xi, \zeta \in \mathbb{R}^k$ with $|\zeta|^2 + |\xi|^2 = 1$ and consider $f := |A\xi + uA\zeta|$ as a function of the argument A. Then f is $\sqrt{2}$-Lipschitz with respect to the Hilbert-Schmidt norm. Therefore Gaussian concentration inequalities imply that the function f must be strongly concentrated around its expected value $\mathbb{E}f$. Specifically, we get for $t > 0$

$$\mathbb{P}(|f - \mathbb{E}f| > t) < 2\exp(-nt^2/4). \tag{4.24}$$

To determine the magnitude of $\mathbb{E}f$, we shall first calculate the second moment.

$$\begin{aligned}
\mathbb{E}f^2 &= \mathbb{E}\,|A\xi + uA\zeta|^2 \\
&= \mathbb{E}\,|A\xi|^2 + \mathbb{E}\,|uA\zeta|^2 + 2\mathbb{E}\langle A\xi, uA\zeta\rangle \\
&= |\xi|^2 + |\zeta|^2 + 2\langle \xi, \zeta\rangle\, \frac{\mathrm{tr}\ u}{n},
\end{aligned}$$

the last equality following, for example, by direct calculation in coordinates. The assumption $|\xi|^2 + |\zeta|^2 = 1$ implies $|\langle \xi, \zeta\rangle| \le 1/2$ and so, recalling our notation $\alpha = \mathrm{tr}\,(\mathrm{Id} - u)/n = 1 - \mathrm{tr}\ u/n$, we deduce that

$$\alpha \le \mathbb{E}f^2 \le 2 - \alpha.$$

Since, by the Khinchine-Kahane inequality, the L_2- and the L_1-norm of a Gaussian vector differ at most by factor $\sqrt{\pi/2}$ (see [LO] for an argment which gives the optimal value of the constant), it follows that

$$\varepsilon_1 := \sqrt{2/\pi}\ \sqrt{\alpha} \le \mathbb{E}f \le \sqrt{2 - \alpha}.$$

Thus choosing $t = \varepsilon_1/3$ in (4.24) yields

$$\mathbb{P}\left(2\varepsilon_1/3 \le |A\xi + uA\zeta| \le \sqrt{2 - \alpha} + \varepsilon_1/3\right) \ge 1 - 2e^{-n\alpha/(18\pi)}. \tag{4.25}$$

The estimates on $|A\xi + uA\zeta|$ and the associated probabilities extend appropriately by homogeneity to any $\xi, \zeta \in \mathbb{R}^k$. The next step is now standard: we choose a proper net in the set $\{\xi, \zeta \in \mathbb{R}^k : |\zeta|^2 + |\xi|^2 = 1\}$ and if the estimates on $|A\xi + uA\zeta|$ hold simultaneously for all elements of that net, it will follow that

$$1/3\sqrt{2/\pi}\ \sqrt{\alpha}\ (|\xi|^2 + |\zeta|^2)^{1/2} \le |A\xi + uA\zeta| \le 2(|\xi|^2 + |\zeta|^2)^{1/2},$$

for all $\xi, \zeta \in \mathbb{R}^k$. The left hand side inequality above yields then the first inequality in (4.6). The right hand side inequality is a statement formally stronger than the second inequality in (4.6).

To conclude the argument we just need to assure the proper resolution of the net and to check its cardinality. Generally, if a linear map is bounded from above by B on an ε-net of the sphere, it is bounded on the entire sphere by $B' = B/(1-\varepsilon)$. If it is additionally bounded on the net from below by b, then it is bounded from below on the entire sphere by $b' = b - B'\varepsilon$. If we choose $\varepsilon = \varepsilon_1/6$, then the resulting B' is < 2, and so $b' > 2\varepsilon_1/3 - 2\varepsilon = \varepsilon_1/3$, as required. Finally, the ε-net can be chosen so that its cardinality is $\leq (1+2/\varepsilon)^{2k} = (1+\sqrt{18\pi/\alpha})^{2k}$, and so the logarithm of the cardinality is $O(k\log(2/\alpha))$. Combining this with (4.25) we obtain an estimate on probability which is exactly of the type asserted in Lemma 4.1. $\square$

Proof of Lemma 4.2 The argument here is similar to that of Lemma 4.1 but substantially simpler since we need only an upper estimate. First, we may assume without loss of generality that T is diagonal. A direct calculation shows then that $\mathbb{E}\,|TA\xi|^2 = (\|T\|_{HS}^2/n)\,|\xi|^2$. Thus, if $|\xi| = 1$, then $\mathbb{E}\,|TA\xi| \leq a/\sqrt{n}$, while the Lipschitz constant of $|TA\xi|$ (in argument A, with respect to the Hilbert-Schmidt norm) is $\leq \|T\|$. It is now enough to choose a 1/2-net on the sphere S^{k-1} and argue as earlier, but paying attention to upper estimates only. $\square$

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